Multiple choice bifurcations as a source of unpredictability in dynamical systems

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Most of the nonlinear systems are characterized by the coexistence of at least two attractors in some regions of the phase space. In this Brief Report, we present an example of a system which exhibits types of bifurcations in which multiple coexisting attractors are created or destroyed simultaneously. The main feature of these bifurcations is that they lead to unpredictable behavior of trajectories when a system parameter is slowly varied through the bifurcation point. [S1063-651X(98)14010-2]

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A great number of dynamical systems with practical importance is characterized by the coexistence of more than two attractors in some regions of parameter space. Nonlinear electrical circuits [1], mechanical buckled beam problems [2], and geophysical models [3] are just simple examples of such systems. Multiple coexisting attractors are common in coupled systems [4,5]. For example, in [6] the problem of synchronization of two identical chaotic systems originally evolving on different coexisting attractors was considered. It has been found that such systems can synchronize on one attractor, but the answer to the question, ''On which one?'' cannot be predicted.

In this Brief Report, we study the bifurcations as the result of which some attractors are destroyed and the system evolution has to switch to one of at least two coexisting attractors. We call them *multiple choice bifurcations* and present examples that such bifurcations are common in two-dimensional piecewise linear systems.

Our main result in this paper is that multiple choice bifurcations are characterized by a new type of sensitivity to noise that can be described in the following way (see Fig. 1). Let the system under consideration be affected by the small noise $\delta x(t)$ (for all $t|\delta x(t)|$ is bound by some small value ϵ), and assume that the system parameter c is varying slowly through the bifurcation point. Before bifurcation the system trajectory evolves on a particular attractor A (or due to noise in its small neighborhood). After a bifurcation at least two attractors B and C exist and attractor A loses its stability (it is transformed into an unstable set located on the boundary of the basins of attraction of attractors B and C), as shown in Fig. 1. We show that the question ''After the bifurcation, on which attractor does the trajectory evolve?'' has no answer, even for infinitely small noise level ϵ.

As an example, consider the dynamics of a four-parameter family of two-dimensional piecewise linear noninvertible map F:

$f_{l,p}(x_n)+d(y_n-x_n)$:

$$x_{n+1}=px_n+\frac{l}{2}\left(1-\frac{p}{l}\right)\left(\left|x_n+\frac{1}{l}\right|-\left|x_n-\frac{1}{l}\right|\right)+d(y_n-x_n),$$

$f_{l,p}(y_n)+d(x_n-y_n)$:

$$y_{n+1}=py_n+\frac{l}{2}\left(1-\frac{p}{l}\right)\left(\left|y_n+\frac{1}{l}\right|-\left|y_n-\frac{1}{l}\right|\right)+d(x_n-y_n), \qquad (1)$$

where $l,p,d\in\mathbb{R}$. Note that this system, which consists of two identical linearly coupled one-dimensional maps, is the generalization of the skew tent map. Chaotic attractors of skew tent maps have been considered in [5]. A map (1) is an example of piecewise smooth systems that are common in number of practical applications [7].

As the first example of multiple choice bifurcation consider map (1) for $l=1.5$, $p=-2.4$. For a small d it has four coexisting attractors $A_{1,2},B,C$ shown in Fig. 2(a) ($d=0.21$). After a bifurcation two of them A_1 and A_2 disappear and trajectories evolving on them go to one of the surviving attractors B or C, shown in Fig. 2(b) ($d=0.25$). As the result of bifurcation the attractors A_1 and A_2 are transformed into the saddle periodic orbits with the stable (along a line $x=-y$) and unstable manifolds on the boundary of the basins of attraction of attractors B and C, as shown in Fig. 2(b). Infinitely small random perturbation into blue (yellow) region direct the trajectory towards attractor $B(C)$.

In the second example let us take $p=-2.4$, $d=0.4$. For $l<l_0=1$ map (1) has a single attractor $x=y=0$ [Fig. 3(a) $l=0.98$.] In the bifurcation at l_0 this attractor is destroyed, two chaotic attractors B and C shown in Fig. 3(b) ($l=1.02$) are born, and the system trajectory from A has to go

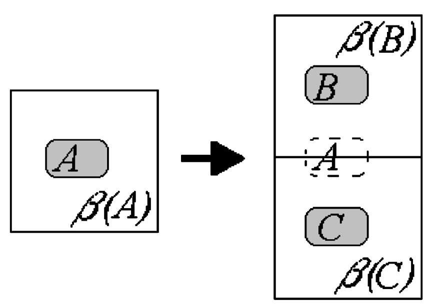

FIG. 1. Multiple choice bifurcation.

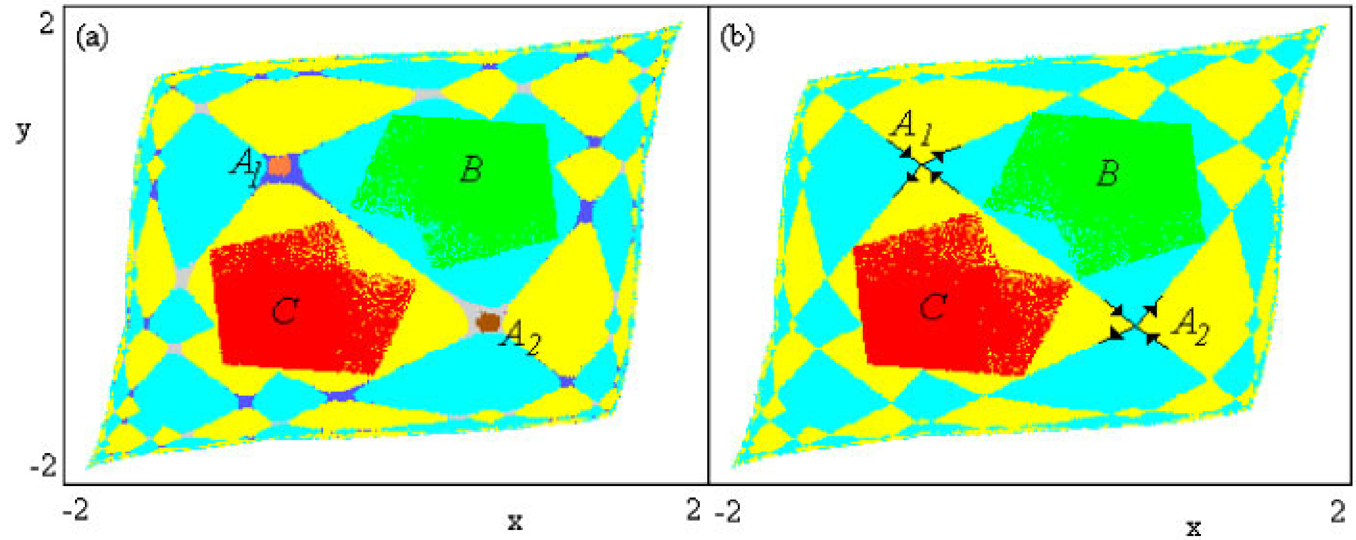

FIG. 2. (Color) Attractors of map (1) and its basins of attraction: $l=1.5$, $p=-2.4$; (a) before multiple choice bifurcation $d=0.21$, (b) after bifurcation $d=0.24$.

to either B or C. Before the bifurcation the system trajectory goes to the attractor along the line $x=y$ and can reach it in two different ways: through positive and negative values of $x=y$. Initial conditions determining positive and negative route are shown in Fig. 3(a) in blue and yellow, respectively. After a bifurcation attractor A is transformed into a periodic saddle with stable (along a line $x=-y$) and unstable (along a line $x=y$) manifolds. Trajectories originating from the neighborhood of this saddle approach attractor B or C along a line $x=y$. Initial conditions which determined the positive and negative route to attractor A now define basins of attraction of attractors B and C. Again the small random perturbation at the bifurcation determines the future attractor of the system trajectory.

Multiple choice bifurcations are common for the considered system (1) as they have been observed for positive measure set of system parameters p,l,d. We found that the trajectories starting from the same initial conditions with the same level of noise (with the same accuracy of numerical calculations) can reach different attractors for slightly different system parameters. Multiple choice bifurcations have been observed in our prevoius studies on two-dimensional piecewise linear systems [5] and in the number of papers on the dynamics of Chua's circuit [1], but up until now the uncertainty that they produced has not been pointed out.

As the result of multiple choice bifurcation when computing the bifurcation diagrams by following the trajectory on attractor A_1 (or A_2) of the first example or on attractor A in the second one in the presence of even very small noise, one can get different results for different noise realizations. This dynamical undecibility connected with multiple choice bifurcations cannot be avoided without appropriate control of the system at the bifurcation (or slightly before). This control is based on the understanding of system dynamics and in most

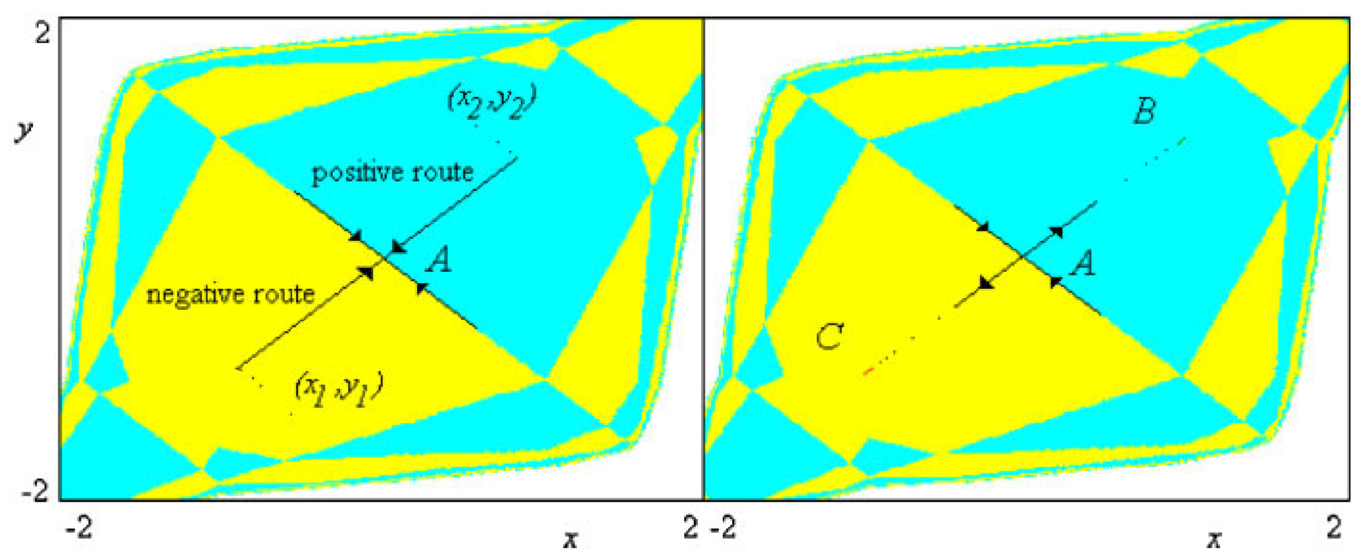

FIG. 3. (Color) Attractors of map (1) and its basins of attraction: $l=1.5$, $d=-0.95$; (a) before multiple choice bifurcation $l=0.98$, (b) after bifurcation $l=1.02$.

systems it can be easily implemented. In our examples it will be based on the deliberate shifting of the system trajectory before bifurcation to the blue or yellow region, depending on which attractor B or C is desired after bifurcation.

It should be noted here that the same type of uncertainty occurs in at least two other cases: (i) when A survives bifurcation but the basins of attraction $\beta(B)$ and $\beta(C)$ of new attractors A and B are infinitely close to the attractor A, as in the case of the transition to riddled basins [4]; (ii) when a periodic attractor A is destroyed in saddle-node bifurcation and a saddle is located on a fractal basin boundary of other two attractors B and C [8]. In both of these cases noise (no matter how small) determines the destination of trajectory that evolved on attractor A before bifurcation. However the origin of uncertainty based on the existence of fractal basin boundaries in the region of the phase space where the destroyed attractor A was located, is somewhat different from our examples where basin boundaries are smooth.

To summarize, we have shown that the bifurcations to multiple attractors in the system with coexisting attractors leads to unpredictable behavior of trajectories as system parameter is slowly varied through its bifurcation value. This dynamical undecidability occurs as the attractor descroyed in bifurcation is located on the basin boundary of attractor are born (or survived) in bifurcation. The described phenomenon seems to be a characteristic of two-dimensional piecewise linear systems. Recently similar results were obtained by Dutta *et al.* [9].

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